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Jackson-type Theorems on Approximation by Trigonometric and Algebraic Pseudopolynomials

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I. INTRODUCTION

The present paper deals with the degree of approximation in $C_{2\pi,2\pi}$ and in $C(I^2)$, the space of continuous and real-valued functions $f(x, y)$ which are 2π-periodic with respect to each variable, and the space of functions which are continuous and real valued on the square I^2 with $I = [-1, +1]$. As approximating functions we shall exclusively deal with blending functions or pseudopolynomials. Functions of this type have already been introduced in A. Marchaud's well-known papers [15, 16], where he considers his pseudopolynomials

$$p\colon [0, 1]^2 \ni (x, y) \mapsto \sum_{i=0}^{m} x^i \cdot A_i(y) + \sum_{j=0}^{n} B_j(x) \cdot y^j \in \mathbb{R};$$

here, A_i and B_j are functions in $B[0, 1]$ (or $B(I)$), the space of bounded, real-valued functions on the interval $[0, 1]$, and $m, n \geqslant 0$ are integers. In case $m = n = 0$ we get $p(x, y) = A(y) + B(x)$, a sum of univariate functions.

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In modern terminology one uses tensor products in order to describe spaces of pseudopolynomials. Given the (real) linear spaces V, W of univariate functions, then the tensor product space $V \otimes W$ of bivariate functions is the span of all product-type functions $f(x)\, g(y)$ with $f \in V$ and $g \in W$. Let P_k and T_k denote the spaces of algebraic and of trigonometric polynomials of degree k or less; thus Marchaud's pseudopolynomials constitute the space $P_m \otimes B(I) + B(I) \otimes P_n$.

In the present paper we shall deal with the corresponding continuous subspace $P_m \otimes C(I) + C(I) \otimes P_n$ of algebraic pseudopolynomials, and with its periodic equivalent, $T_m \otimes C_{2\pi} + C_{2\pi} \otimes T_n$, of trigonometric pseudopolynomials.

The interest in finding approximations from these and related spaces (mainly for the purpose of surface fitting and related computer aided geometric design applications) was revitalized by two fundamental papers of W. J. Gordon [7, 8]. He used Boolean sums or blending methods in order to generate his "hyperpolynomials" or "blending functions." However, a number of little-known papers on these approximation problems had been published earlier in the European literature. In addition to Marchaud's papers, we mention the articles of L. Neder [17] and D. D. Stancu [21]. For more complete historical information the reader is referred to [7, 21].

While the classical papers mainly focused on modifications of interpolation operators (see the work of the Romanian school, in particular), it is known from a paper of M. Nicolescu [18] that the Weierstrass approximation theorem for so-called Bögel continuous functions holds for approximation by certain algebraic pseudopolynomials (see, e.g., I. Badea [2]). More recently, the question whether certain sums of tensor product spaces are proximinal has been dealt with to some extent; here we refer to the survey article of E. W. Cheney [4]. However, there seems to be little information on quantitative assertions. As exceptions we mention I. Badea's [1] quantitative version of Nicolescu's theorem by using Boolean sums of univariate Bernstein operators, and the recent paper of W. Haussmann *et al.* [9], who derive the Favard type estimate for trigonometric blending approximation. We also refer to the paper of W. Haussmann and K. Zeller [10] and to the references therein, where estimates can be found concerning some special situations.

The aim of the present paper is thus to improve and generalize these quantitative assertions by showing that full analogies of S. B. Stečkin's [22] and Yu. A. Brudnyĭ's [3] famous univariate results of the Jackson type hold for approximation by pseudopolynomials as well. These estimates will be given in terms of a certain product-type modulus of continuity $\omega_{k,l}$, which seems to have been used for the first time by A. Marchaud in his 1924 paper. This modulus is sometimes also denoted as the mixed modulus of smoothness (A. F. Timan [23, p. 113]) or as the (k, l)-modulus of smoothness (L. L. Schumaker [19, p. 516]).

Section II of the paper reviews some of the properties of this mixed modulus of smoothness. In Section III, our Jackson-type theorem for the periodic case is presented; Section IV deals with the algebraic setting.

As for the methods employed in this paper, we shall be exclusively working with univariate linear operators L, L' and with the Boolean sum

$${}_xL \oplus {}_yL' = {}_xL + {}_yL' - {}_xL \circ {}_yL'$$

of their parametric extensions ${}_xL$ and ${}_yL'$; here ${}_xL$ acts on the bivariate function $f(x, y)$ as if y is considered to be a fixed parameter, and ${}_yL'$ is described in a similar way. The methods extend to the multivariate case, but we shall not stress this.

II. Remarks on Marchaud's Product-type Modulus

Univariate differences are used in order to define the mixed modulus of smoothness. For a function $f\colon \mathbb{R} \to \mathbb{R}$, let

$$\begin{aligned} \Delta_\partial^0 f(x) &:= f(x), \\ \Delta_\partial^1 f(x) &:= \Delta_\partial f(x) := f(x) - f(x+\partial), \qquad (2.1)\\ \Delta_\partial^k f(x) &:= \Delta_\partial^1(\Delta_\partial^{k-1} f(x)), \qquad 1 < k \in \mathbb{N}, \end{aligned}$$

where $x, \partial \in \mathbb{R}$, and explicitly,

$$\Delta_\partial^k f(x) = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} f(x + \nu\partial), \qquad k \geqslant 0. \tag{2.2}$$

For $f \in C_{2\pi}$ (continuous 2π-periodic functions) the kth order modulus of smoothness of f is defined by

$$\omega_k(f; \varepsilon) := \sup\{\sup\{|\Delta_\partial^k f(x)| : |\partial| \leqslant \varepsilon\} : x \in \mathbb{R}\}, \quad 0 \leqslant \varepsilon \in \mathbb{R}. \tag{2.3}$$

For bivariate functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, we consider the parametric extensions ${}_x\Delta_\partial^h$ and ${}_y\Delta_\eta^k$ of Δ_∂^h and Δ_η^k (acting on the first variable x and the second variable y, respectively). Then, for arbitrary $h, k \in \mathbb{N}_0$ and $\partial, \eta \in \mathbb{R}$ we define product-type differences by

$$\begin{aligned} \Delta_{\partial,\eta}^{h,k} f(x, y) &:= {}_x\Delta_\partial^h[{}_y\Delta_\eta^k f(x, y)) \\ &= \sum_{p=0}^{h} \sum_{\nu=0}^{k} (-1)^{p+\nu} \binom{h}{p}\binom{k}{\nu} f(x + p\partial, y + \nu\eta). \end{aligned} \tag{2.4}$$

The product-type modulus is then for $0 \leqslant \varepsilon \in \mathbb{R}$, $0 \leqslant \rho \in \mathbb{R}$ given by

$$\omega_{h,k}(f; \varepsilon, \rho) := \sup \{\sup \{|\Delta_{\partial,\eta}^{h,k} f(x, y)| : |\partial| \leqslant \varepsilon, |\eta| \leqslant \rho\} : (x, y) \in \mathbb{R}^2\}. \quad (2.5)$$

Badea [1] considers the case $h = k = 1$ in order to define his so-called B-modulus and to give the quantitative version of Nicolescu's theorem mentioned above. Here, $\Delta_{\partial,\eta}^{1,1} f(x, y) = f(x, y) - f(x + \partial, y) - f(x, y + \eta) + f(x + \partial, y + \eta)$. In case $h = k = 0$ we also write $\omega_{0,0}(f; \cdot, *) = \|f\|_\infty$ since, if f is continuous and bounded on $\mathbb{R}^2$, this is the usual Chebyshev norm.

Let us quote some important properties which follow readily from (2.4):

(a) If f is the product of two univariate functions, $f(x, y) = f_1(x) f_2(y)$, then

$$\begin{aligned} \Delta_{\partial,\eta}^{h,k} f(x, y) &= \Delta_\partial^h f_1(x)\, \Delta_\eta^k f_2(y), \\ \omega_{h,k}(f; \varepsilon, \rho) &= \omega_h(f_1; \varepsilon)\, \omega_k(f_2; \rho). \end{aligned} \quad (2.6)$$

As a consequence, if $f_1 \in P_{h-1}$ or $f_2 \in P_{k-1}$, then $\Delta_{\partial,\eta}^{h,k} f = 0$. Thus, for all $0 \leqslant \varepsilon \in \mathbb{R}$, $0 \leqslant \rho \in \mathbb{R}$, the modulus $\omega_{h,k}(\cdot; \varepsilon, \rho)$ annihilates the space $P_{h-1} \otimes V + V \otimes P_{k-1}$ with $V := \{f : \mathbb{R} \to \mathbb{R}\}$.

(b) If we consider $\omega_{h,k}(\cdot; \varepsilon, \rho)$, for fixed ε, ρ, as a functional on the space $C_{2\pi,2\pi}$, then $\omega_{h,k}(\cdot; \varepsilon, \rho)$ has the properties of a semi-norm. Moreover, for $f \in C_{2\pi,2\pi}$ we have by a compactness argument that

$$\omega_{h,k}(f; \varepsilon, \rho) = |\Delta_{\partial^*,\eta^*}^{h,k} f(x^*, y^*)| \quad (2.7)$$

for some appropriate (x^*, y^*), $(\partial^*, \eta^*) \in \mathbb{R}^2$ with $|\partial^*| \leqslant \varepsilon$, $|\eta^*| \leqslant \rho$, and

$$\lim_{(\varepsilon,\rho) \to (0,0)} \omega_{h,k}(f; \varepsilon, \rho) = 0. \quad (2.8)$$

(c) For $f \in C^{p,q}$ (i.e., $f : \mathbb{R}^2 \to \mathbb{R}$ having continuous derivatives $f^{(\alpha,\beta)}$, $0 \leqslant \alpha \leqslant p$, $0 \leqslant \beta \leqslant q$) we get

$$\omega_{p+h,q+k}(f; \varepsilon, \rho) \leqslant \varepsilon^p \rho^q \omega_{h,k}(f^{(p,q)}; \varepsilon, \rho), \quad (2.9)$$

and in particular for $h = k = 0$:

$$\omega_{p,q}(f; \varepsilon, \rho) \leqslant \varepsilon^p \rho^q \|f^{(p,q)}\|_\infty. \quad (2.10)$$

This may be verified by introducing the B-spline functions (cf. L. L. Schumaker [19, p. 517]). For example, if $p > 0$, then

$$B_{p,\partial}(\xi) = 1/(p-1)! \sum_{v=0}^{p} (-1)^v \binom{p}{v} (\xi + v\partial)_+^{p-1}$$

has compact support $[0, -p\partial]$ or $[-p\partial, 0]$ for $\partial < 0$ and $\partial > 0$, respectively, and using integration by parts, we find the convolution formula that, for $f \in C^p(\mathbb{R})$,

$$\Delta_\partial^p f(x) = \int_{-\infty}^{\infty} f^{(p)}(s)\, B_{p,\partial}(x-s)\, ds$$

$$= \int_{-\infty}^{\infty} f^{(p)}(x-s)\, B_{p,\partial}(s)\, ds.$$

The corresponding bivariate convolution formula for $p, q > 0$ is

$$\Delta_{\partial,\eta}^{p,q} f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^{(p,q)}(x-s, y-t)\, B_{p,\partial}(s)\, B_{q,\eta}(t)\, ds\, dt.$$

Hence

$$\begin{aligned} &\Delta_{\partial,\eta}^{p+h,q+k} f(x, y) \\ &\quad = \Delta_{\partial,\eta}^{h,k}(\Delta_{\partial,\eta}^{p,q} f(x, y)) \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} {}_x\Delta_\partial^h\, {}_y\Delta_\eta^k\, f^{(p,q)}(x-s, y-t)\, B_{p,\partial}(s)\, B_{q,\eta}(t)\, ds\, dt, \end{aligned}$$

and for $|\partial| \leqslant \varepsilon$, $|\eta| \leqslant \rho$

$$\begin{aligned} &|\Delta_{\partial,\eta}^{p+h,q+k} f(x, y)| \\ &\quad \leqslant \omega_{h,k}(f^{(p,q)}; \varepsilon, \rho) \cdot \left| \int_{-\infty}^{\infty} B_{p,\partial}(s)\, ds \int_{-\infty}^{\infty} B_{q,\eta}(t)\, dt \right| \\ &\quad = |\partial|^p \cdot |\eta|^q \cdot \omega_{h,k}(f^{(p,q)}; \varepsilon, \rho). \end{aligned}$$

Now the estimate given in (2.9) is apparent.

III. Jackson-type Theorems for Trigonometric Blending Approximation

Jackson's theorem in its general form tells us that, for every $h, m \in \mathbb{N}$, there is a linear operator

$$J_{h,m}: C_{2\pi} \to T_m \tag{3.1}$$

satisfying

$$\|f - J_{h,m}(f)\|_\infty \leqslant M_h \omega_h(f; 1/m) \tag{3.2}$$

for some constant M_h independent of f and m. In this section we shall extend this by proving the following bivariate result.

THEOREM 3.1. *For every h, k, m, $n \in \mathbb{N}$ there is a linear operator*

$$J = J_{h,m;k,n}: C_{2\pi,2\pi} \to T_m \otimes C_{2\pi} + C_{2\pi} \otimes T_n$$

satisfying the estimate

$$\|f - J(f)\|_\infty \leqslant M_h M_k \omega_{h,k}(f; 1/m, 1/n). \tag{3.3}$$

Proof. We may assume that the operator $J_{h,m}$ in (3.1) is defined by some higher-order Jackson kernel

$$u_{h,m}(t) = 1 + 2 \sum_{i=1}^{m} a_i \cos it \geqslant 0$$

with real coefficients $a_i = a_{i,h,m}$ and such that

$$[f - J_{h,m}(f)](x) = 1/2\pi \int_{-\pi}^{\pi} \Delta_t^h f(x)\, u_{h,m}(t)\, dt,\ f \in C_{2\pi} \tag{3.4}$$

(cf., e.g., Lorentz [14, p. 57]). Theorem 3.1 will be proved by using the Boolean sum $J = {}_xJ_{h,m} \oplus {}_yJ_{k,n}$. Indeed, if we put

$$[f - J(f)](x, y) := 1/4\pi^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Delta_{t,s}^{h,k} f(x,y)\, u_{h,m}(t)\, u_{k,n}(s)\, dt\, ds \tag{3.5}$$

for $f \in C_{2\pi,2\pi}$, then

$$\begin{aligned}
&[f - J(f)](x, y) \\
&\quad = f(x, y) - {}_xJ_{h,m} f(x, y) - {}_yJ_{k,n} f(x, y) + {}_xJ_{h,m} \circ {}_yJ_{k,n} f(x, y) \\
&\quad = (F - {}_xJ_{h,m} F)(x, y).
\end{aligned}$$

where $F(x, y) = (f - {}_yJ_{k,n} f)(x, y)$.

Now, using (3.2) for the operators $J_{h,m}$ and $J_{k,n}$ and some properties of the univariate modulus of continuity, we get for appropriate (x_0, y_0), (x^*, y^*), and $(\partial^*, \eta^*) \in \mathbb{R}^2$ with $|\partial^*| \leqslant 1/m$, $|\eta^*| \leqslant 1/n$:

$$\begin{aligned}
\|f-J(f)\|_\infty &= |(F-{}_xJ_{h,m}F)(x_0, y_0)| \\
&= \max\{|(F-{}_xJ_{h,m}F)(x, y_0)|: x\in\mathbb{R}\} \\
&\leqslant M_h\omega_h(F(\cdot, y_0); 1/m) \\
&= M_h|{}_x\Delta^h_{\partial^*}F(x^*, y_0)| \\
&= M_h|({}_x\Delta^h_{\partial^*}f - {}_yJ_{k,n}[{}_x\Delta^h_{\partial^*}f])(x^*, y_0)| \\
&\leqslant M_h \max\{|({}_x\Delta^h_{\partial^*}f - {}_yJ_{k,n}[{}_x\Delta^h_{\partial^*}f])(x^*, y)|: y\in\mathbb{R}\} \\
&\leqslant M_hM_k\omega_k({}_x\Delta^h_{\partial^*}f(x^*, \cdot); 1/n) \\
&= M_hM_k|{}_y\Delta^k_{\eta^*}({}_x\Delta^h_{\partial^*}f(x^*, y^*))| \\
&\leqslant M_hM_k\omega_{h,k}(f; 1/m, 1/n). \quad \blacksquare
\end{aligned}$$

Some consequences may be listed.

COROLLARY 3.2. *For $f\in C_{2\pi,2\pi}$ and $V := T_m\otimes C_{2\pi} + C_{2\pi}\otimes T_n$ we have*

$$\inf\{\|f-v\|_\infty : v\in V\} \leqslant M_hM_k\omega_{h,k}(f; 1/m, 1/n) \tag{3.6}$$

for arbitrary $h, k\in\mathbb{N}$.

COROLLARY 3.3. *For $f(x, y) = f_1(x)\,f_2(y)$ with $f_1, f_2\in C_{2\pi}$ we have*

$$\min\{\|f-v\|_\infty : v\in V\} \leqslant M_h\cdot\omega_h(f_1; 1/m)\cdot M_k\cdot\omega_k(f_2; 1/n). \tag{3.7}$$

This estimate is consistent with the fact that $\min\{\|f-v\|_\infty : v\in V\} = \|f-v^*\|_\infty$ with $(f-v^*)(x, y) = [(f_1-v_1)(x)]\cdot[(f_2-v_2)(y)]$, where $v_1\in T_m$ and $v_2\in T_n$ are univariate best Chebyshev approximations to f_1 and f_2, respectively (Haussmann and Zeller [11]).

The following are obtained through the observations made at the end of Section II.

COROLLARY 3.4. *For $f\in C^{p,q}_{2\pi,2\pi}$ and $V = T_m\otimes C_{2\pi} + C_{2\pi}\otimes T_n$ we have*

$$\inf\{\|f-v\|_\infty : v\in V\} \leqslant M_pM_qm^{-\alpha}n^{-\beta}\omega_{p-\alpha,q-\beta}(f^{(\alpha,\beta)}; 1/m, 1/n),$$
$$0\leqslant\alpha\leqslant p,\ 0\leqslant\beta\leqslant q. \tag{3.7}$$

In case $\alpha = p$, $\beta = q$ this reads

$$\inf\{\|f-v\|_\infty : v\in V\} \leqslant M_pM_qm^{-p}n^{-q}\|f^{(p,q)}\|_\infty. \tag{3.7'}$$

Remark 3.5. (i) Estimate (3.7′) yields the same order of approximation for differentiable functions as given in the paper of Haussman *et al.* [9], where, in addition, best possible values for the constants M_p and M_q can be found (Favard constants κ_p and κ_q, respectively). The same estimate is also obtained by applying a recent rather general, but non-constructive argument of M. v. Golitschek [5].

(ii) The problem of finding good (or even best possible) constants $M_h \cdot M_k$ in Theorem 3.1 and its Corollary 3.2 is far more involved and requires considerably more information on the univariate case than was used in the proof of Theorem 3.1. To our knowledge it is not known, for instance, what the best possible functions $c_k(\gamma)$, $k \geqslant 1$, $\gamma > 0$, are in an estimate of type

$$\inf\{ \|f - p_m\|_\infty : p_m \in T_m\} \leqslant c_k(\gamma) \cdot \omega_k(f; \gamma/m), \, m \geqslant 1.$$

With the exception of the case $k = 1$ (where the well-known result of Korneičuk is available) an analogous statement is true for estimates in terms of the more common quantity $\omega_k(f; \pi/(m+1))$, $m \geqslant 0$. However, a significant contribution in this direction (for the case $k = 2$ and by using linear methods) was made by V. V. Žuk [24]; his results will be generalized to the cases $k > 2$ in a forthcoming paper of A. Sperling and the first author of this article.

IV. The Algebraic Case

We also find pointwise Jackson-type estimates for approximating functions $f \in C(I^2)$, $I = [-1, 1]$, by elements of the algebraic blending space

$$W := P_m \otimes C(I) + C(I) \otimes P_n, \qquad m, n \geqslant 1. \tag{4.1}$$

Here the modulus (2.5) has to be slightly modified, namely

$$\omega_{h,k}(f; \varepsilon, \rho) := \sup |\Delta_{\partial,\eta}^{h,k} f(x, y)|$$

where the sup is now taken over all $(x, y) \in I^2$, $(\partial, \eta) \in \mathbb{R}^2$ such that $(x + h\delta, y + k\eta) \in I^2$, $|\partial| \leqslant \varepsilon$, and $|\eta| \leqslant \rho$.

For the case $h = k = 1$ it is possible to arrive at a Jackson-type theorem using the trigonometric transformation technique known from the univariate case. However, it is known from this case as well that this technique fails for the higher-order cases. This is our reason for using a different and even simpler approach.

As was the case for the above $C_{2\pi,2\pi}$ theorem, our result will be based upon an estimate for approximation of univariate functions by certain linear operators.

THEOREM (see H. Gonska and E. Hinnemann [6]). *Let $r \geqslant 0$ and $s \geqslant 1$. Then there is a sequence $Q_n = Q_n^{(r,s)}$ of linear polynomial operators mapping $C^r(I)$ into P_n, such that for all $f \in C^r(I)$, all $|x| \leqslant 1$ and all $n \geqslant \max(4(r+1), r+s)$ one has*

$$|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqslant M_{r,s} \cdot \Delta_n(x)^{r-k} \cdot \omega_s(f^{(r)}; \Delta_n(x)), \quad 0 \leqslant k \leqslant r.$$

Here $\Delta_n(x) := (1-x^2)^{1/2} \cdot n^{-1} + n^{-2}$, and $M_{r,s}$ is a constant independent of f, x, and n.

For our purposes it suffices to use the operators $Q_n^{(0,s)}$, which we shall denote by $Q_{s,n}$ for the sake of brevity. We also put $M_s := M_{0,s}$.

THEOREM 4.1. *For every $h, k \in \mathbb{N}$ and $m \geqslant \max(4, h)$, $n \geqslant \max(4, k)$ there is a linear operator $Q = Q_{h,m;k,n}$: $C(I^2) \to P_m \otimes C(I) + C(I) \otimes P_n$ satisfying the estimate*

$$|f(x, y) - Q(f; x, y)| \leqslant M_h \cdot M_k \cdot \omega_{h,k}(f; \Delta_m(x), \Delta_n(y)), \quad (x, y) \in I^2.$$

The constants M_h and M_k are given as in the preceding theorem, and are thus independent of f, (x, y), and (m, n).

Proof. As we did in the proof of Theorem 3.1, we define Q to be the Boolean sum of (parametric extensions of) univariate operators. More precisely,

$$Q := {}_xQ_{h,m} \oplus {}_yQ_{k,n} = {}_xQ_{h,m} + {}_yQ_{k,n} - {}_xQ_{h,m} \circ {}_yQ_{k,n},$$

and

$$[f - Q(f)](x, y) = (F - {}_xQ_{h,m}F)(x, y)$$

with $F(x, y) = (f - {}_yQ_{k,n}f)(x, y)$.

Now

$$\begin{aligned}
|[f - Q(f)](x, y)| &= |(F - {}_xQ_{h,m}F)(x, y)| \\
&\leqslant M_h \omega_h(F(\cdot, y); \Delta_m(x)) \\
&= M_h |{}_x\Delta_{\partial^*}^h F(x^*, y)| \qquad \text{with} \quad |\partial^*| \leqslant \Delta_m(x) \\
&= M_h |{}_x\Delta_{\partial^*}^h f(x^*, y) - {}_yQ_{k,n}({}_x\Delta_{\partial^*}^h f(x^*, y))| \\
&\leqslant M_h M_k \omega_k({}_x\Delta_{\partial^*}^h f(x^*, \cdot); \Delta_n(y)) \\
&= M_h M_k \, {}_y\Delta_{\eta^*}^k({}_x\Delta_{\partial^*}^h f(x^*, y^*)) \qquad \text{with} \quad |\eta^*| \leqslant \Delta_n(y) \\
&\leqslant M_h M_k \omega_{h,k}(f; \Delta_m(x), \Delta_n(y)). \quad \blacksquare
\end{aligned}$$

The following corollaries are analogous to the ones obtained for the trigonometric case

COROLLARY 4.2. *For* $f \in C(I^2)$ *and* $V := P_m \otimes C(I) + C(I) \otimes P_n$, $(m, n) \geq (\max(4, h), \max(4, k))$, one has

$$\inf\{\|f - v\|_\infty : v \in V\} \leq M_h \cdot M_k \cdot \omega_{h,k}(f; 2/n, 2/m)$$

for $h, k \geq 1$.

COROLLARY 4.3. *For* $f(x, y) = f_1(x) \cdot f_2(y)$, $f_1, f_2 \in C(I)$, *and* $(m, n) \geq (\max\{4, h\}, \max\{4, k\})$ *one obtains the inequality*

$$\min\{\|f - v\|_\infty : v \in V\} \leq M_h \cdot \omega_h(f_1; 2/m) \cdot M_k \cdot \omega_k(f_2; 2/n).$$

COROLLARY 4.4. *For any* $f \in C^{p,q}(I^2)$ *there is an element* $v \in P_m \otimes C(I) + C(I) \otimes P_n$ *such that for all* $(x, y) \in I^2$ *and* $0 \leq \alpha \leq p$, $0 \leq \beta \leq q$ *there holds*:

$$|f(x, y) - v(x, y)| \leq M_p \Delta_m^\alpha(x) \cdot M_q \Delta_n^\beta(y) \cdot \omega_{p-\alpha, q-\beta}(f^{(\alpha,\beta)}; \Delta_m(x), \Delta_n(y)).$$

For the case $\alpha = p$, $\beta = q$ *we have*

$$|f(x, y) - v(x, y)| \leq M_p \cdot \Delta_m^p(x) \cdot M_q \cdot \Delta_n^q(y) \cdot \|f^{(p,q)}\|_\infty.$$

Remark 4.5. The determination of best possible constants M_h for univariate results is even more complicated in the algebraic case. For instance, the proof of Theorem 1 in Haussmann *et al.* [9] makes heavy use of the Favard–Ahiezer–Krein theorem, but no complete analogy of this theorem is known for the algebraic case, although an important contribution is due to H. Sinwel [20].

Pointwise estimates in terms of first- and higher-order moduli of smoothness of arbitrary continuous functions exhibiting small values of the constants are even more difficult to achieve. For the univariate case and $h = 1$ we mention papers of O. Kiš and Ho Tho Cao [12], who obtained small constants via an interpolatory approach of Lagrange type, and of H. G. Lehnhoff [13], who investigated an efficient sequence of univariate positive linear operators.

Note added in proof. It has been brought to our knowledge through Yu. A. Brudnyĭ's recent review No. 537.41012 in "Zentralblatt für Mathematik" (March 1985) that he as well as M. K. Potapov have dealt with questions related to our paper (*Izv. Akad. Nauk SSSR, Ser. Mat.* **34** (1970), 564–583, and *Tr. Mat. Inst. Steklova* **117** (1972), 256–291, respectively). However, both authors treat the L^p case, $1 \leq p \leq \infty$, while our paper deals with continuous pseudopolynomials. Furthermore, neither of their papers contain pointwise assertions.

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